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Continuous-time stochastic consensus: Stochastic approximation and Kalman–Bucy filtering based protocols[☆]

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ABSTRACT

This paper investigates the continuous-time multi-agent consensus with stochastic communication noises. Each agent can only use its own and neighbors' information corrupted by random noises to design its control input. To attenuate the communication noises, we consider the stochastic approximation type and the Kalman–Bucy filtering based protocols. By using the tools of stochastic analysis and algebraic theory, the asymptotic properties of these two kinds of protocols are analyzed. Firstly, for the stochastic approximation type protocol, we clarify the relationship between the convergence rate of the consensus error and a representative class of consensus gains in both mean square and probability one. Secondly, we propose Kalman–Bucy filtering based consensus protocols. Each agent uses Kalman–Bucy filters to get asymptotically unbiased estimates of neighbors' states and the control input is designed based on the protocol with precise communication and the certainty equivalence principle. The iterated logarithm law of estimation errors is developed. It is shown that if the communication graph has a spanning tree, then the consensus error is bounded above by $O(t^{-1})$ in mean square and by $O(t^{-1/2}(\log\log t)^{1/2})$ almost surely. Finally, the superiority of the Kalman–Bucy filtering based protocol over the stochastic approximate type protocol is studied both theoretically and numerically.

1. Introduction

In recent years, the distributed coordination of multi-agent systems (MASs) has attracted more and more attention of multi-disciplinary researchers, due to its wide applications in the formation control (Fax & Murray, 2004), the distributed optimization (Nedic & Ozdaglar, 2009), and the flocking problem (Olfati-Saber, 2006). The consensus problem, which is one of the most fundamental topics in the distributed coordination, has been widely studied in the system and control community motivated by Vicsek's model in Vicsek, Czirok, Ben-Jacob, Cohen, and Sochet (1995). Consensus control generally means to design a distributed protocol such that all agents asymptotically reach an agreement on their states. A comprehensive survey on consensus problems can be found in Ren, Beard, and Atkins (2005) and more recent results can be found in Nourian, Caines, and Malhame (2014), Pasqualetti, Borra, and Bullo (2014) and Su and Huang (2012), etc.

Consensus problems with random measurement or communication noises have attracted several researchers since such modeling reflects many practical properties of distributed networks. For the consensus protocol with precise communication, for a given agent, the weighted sum of relative states between neighbors and itself is used to update the agent's state. This weighted sum of relative states can be viewed as a kind of spacial innovation. For the case with communication noises, the spacial innovation is corrupted. To attenuate the noise effect, one idea is using the cautious control, that is, to decrease the algorithm gain. As long as the consensus system evolves, the differences between agents' states become smaller and smaller, then the new information contained in the space innovation corrupted by noises becomes less and less, so a vanishing algorithm gain has to be used. This is so called distributed stochastic approximation type consensus.

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Huang and Manton (2009) considered the discrete-time stochastic approximation type consensus algorithm with fixed topologies. They proved that if the consensus gain $a(k)$ (where k is the discrete time instant) decays with a rate $O(1/k^{\gamma})$, $\gamma \in (0.5, 1]$, the communication graph has the circulant invariance property and strong connectivity, then the algorithm ensures both mean square and almost sure consensus. Huang and Manton (2010) extended the results to the case with general digraphs, which proved that if the digraph contains a spanning tree, then $\sum_{k=0}^{\infty} a(k) = \infty$ and $\sum_{k=0}^{\infty} a^2(k) < \infty$ suffices for both mean square and almost sure consensus. Kar and Moura (2010) considered the discrete-time distributed averaging with quantized data and random link failures. By using dithered quantization, the quantization error sequence is transformed to white noises and the stochastic approximation consensus protocol is employed to ensure mean square convergence of the algorithm. Li and Zhang (2010) considered the case of fixed and time-varying topologies, and they showed if the network switches between jointly-containing-spanning-tree, instantaneously balanced digraphs, then the designed protocol can guarantee that each individual state converges, both almost surely and in mean square, to a common random variable, whose expectation is right the average of the initial states of the whole system. Besides, a rough estimate of the almost sure convergence rate for the consensus error was given. Continuous-time stochastic approximation type consensus problems have also been widely studied. Li and Zhang (2009) showed that if the network is a balanced digraph containing a spanning tree, then a necessary and sufficient condition to guarantee the asymptotic unbiased mean square average-consensus is $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} a^2(t)dt < \infty$. More extended results on continuous-time stochastic approximation type consensus protocols can be found for the leader-following cases (Hu & Feng, 2010; Ma, Li, & Zhang, 2010), the case with general digraphs (Wang & Zhang, 2009), the case with time-delay (Liu, Liu, Xie, & Zhang, 2011) and the cases of second-order and linear dynamics with static state feedback (Cheng, Hou, & Tan, 2014; Cheng, Hou, Tan, & Wang, 2011). And recently, this kind of protocols are applied to the containment control of multi-agent systems with random measurement noises (Wang, Cheng, Hou, Tan, & Wang, 2014).

The works on continuous-time stochastic approximation type consensus protocol mainly concentrated on the conditions to ensure the mean square or almost sure consensus. However, its asymptotic convergence rate, which represents the negotiation speed of the agents as time goes to infinity, is rarely investigated in the relevant literature. It is more meaningful to study the relationship among the asymptotic convergence rate, the consensus gain function $a(t)$, and the communication graph. In this paper, motivated by the above discussions, we investigates the asymptotic convergence rate of the continuous-time stochastic approximation type consensus protocol. We consider a representative class of consensus gains which satisfy $\int_0^{\infty} a(t)dt = \infty$ and $\int_0^{\infty} a^2(t)dt < \infty$. Using the basic results of stochastic analysis and algebraic graph theory, in particular the law of the iterated logarithm of stochastic integrals, we get precise estimations of the convergence rate of the consensus error. It is found that if the consensus gain satisfies that $\lim_{t\to\infty}(t^{\gamma}a(t))$ exists and is positive for $\gamma \in (0.5, 1]$, and for $\gamma = 1$, $\lim_{t\to\infty}(ta(t)) > 1/(2\lambda_{\min})$, with $\lambda_{\min}$ denoting the smallest real part of Laplacian eigenvalues of the network graph, we have: (i) the mean square of the consensus error is bounded above by $O(t^{-\gamma})$ asymptotically; (ii) for the case with balanced digraphs, the mean square of the consensus error is bounded both above and below by $\Theta(t^{-\gamma})$ asymptotically; (iii) the consensus error is almost surely bounded above by $O(t^{-\gamma/2+\varepsilon})$, $\forall\, \varepsilon > 0$, asymptotically for the case with undirected graphs. In this paper, we improve the results of Li and Zhang (2009, 2010) in the sense that (i) the mean square convergence rate of the continuous-time stochastic approximation type consensus protocol is first given; (ii) the almost sure convergence rate is estimated more precisely.

Since the vanishing consensus gain function is used in the stochastic approximation type consensus protocol, the communication noises are attenuated at the price of a slower convergence rate of the algorithm (Huang & Manton, 2009). This motivates us to propose another idea to attenuate the communication noises. The received information from neighbors can be filtered firstly to get the estimates of neighbors' states, then the estimates can be used instead of the true states for the control protocol design. This methodology for controller design is often used in single-agent control systems and is called the certainty equivalence principle. It is well-known that the Kalman–Bucy filter is the main tool of state estimation for continuous-time linear systems driven by Gaussian white noises (Kallianpur, 1980; Øksendal, 2010). Here, based on the Kalman-Bucy filtering theory, we design a filter for each noisy communication link to get the asymptotically unbiased estimates of neighbors' states, then the control input of each agent is designed based on the consensus protocol with precise communication (Olfati-Saber & Murray, 2004) and the certainty equivalence principle. We develop the iterated logarithm law of estimation errors and show that if the communication graph has a spanning tree, then this novel Kalman-Bucy filtering based protocol leads to both mean square and almost sure weak consensus. Moreover, the mean square of the consensus error is bounded above by $O(t^{-1})$ asymptotically, and the consensus error for each agent is almost surely bounded above by $O(t^{-1/2}(\log\log t)^{1/2})$. Comparing the convergence rates of these two kinds of protocols, it is shown that the Kalman-Bucy filtering based protocol leads to a higher convergence rate than the stochastic approximation type protocol in some circumstances. Especially, we verify this superiority for the case with undirected graphs.

The paper is organized as follows. In Section 2, we formulate the considered consensus problem. Section 3 gives the asymptotic convergence properties for the stochastic approximation type protocol. In Section 4, we introduce the Kalman-Bucy filtering based consensus protocol and analyze the convergence rate of the consensus error. The protocol is also applied to a leader-following scenario. The asymptotic convergence properties of these two kinds of protocols are compared in Section 5, while a numerical example is presented. Finally, Section 6 concludes the paper and gives some interesting future topics.

In this paper, we will adopt the following notations: $\mathscr{R}^{m\times n}$ denotes the $m \times n$ dimensional real space; $\mathbf{1_{N\times 1}}$ denotes an $N \times 1$ column vector with all ones; $\mathbf{0_{N\times 1}}$ denotes an $N \times 1$ column vector with all zeros. For a given vector or matrix A, A^T denotes its transpose, and $\|A\|$ denotes its Frobenius norm. For a given complex number λ, $Re(\lambda)$ denotes its real part and $Im(\lambda)$ denotes its imaginary part. The notion $f(t) = O(g(t))$ denotes $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$; $f(t) = \Omega(g(t))$ denotes $\liminf_{t\to\infty} |f(t)/g(t)| > 0$; $f(t) = \Theta(g(t))$ denotes $0 < \liminf_{t\to\infty} |f(t)/g(t)| \le \limsup_{t\to\infty} |f(t)/g(t)| < \infty$ and $f(t) = o(g(t))$ denotes $\lim_{t\to\infty} |f(t)/g(t)| = 0$.

2. Problem formulation

Let the communication topology of MASs be modeled by a weighted digraph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$. The set of nodes $\mathscr{V} = \{1, \ldots, N\}$, and node i represents the ith agent. A pair (j, i) belongs to the edge set $\mathscr{E} \Leftrightarrow$ the jth agent can send information to the ith agent directly. Here, j is called the parent of i, and i is called the child of j. The neighborhood of the ith agent is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} | (j, i) \in \mathscr{E}\}$. Node i is called a source if it has no parent but only children. The weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathscr{R}^{N\times N}$. For any $i, j \in \mathscr{V}$, $a_{ij} \ge 0$, and $a_{ij} > 0 \Leftrightarrow j \in \mathscr{N}_i$. The Laplacian matrix $L_{\mathscr{G}} = \mathscr{D} - \mathscr{A}$, where $\mathscr{D} = \mathrm{diag}(\sum_{j=1}^{N} a_{1j}, \ldots, \sum_{j=1}^{N} a_{Nj})$.

The digraph $\mathscr{G}$ is called a balanced digraph, if $\sum_{j=1}^{N} a_{ji} = \sum_{j=1}^{N} a_{ij}$, $i = 1, 2, \ldots, N$. It is clear that an undirected graph is a balanced digraph. A directed tree is a digraph, where every node except the root has exactly one parent and the root is a source. A spanning tree of $\mathscr{G}$ is a directed tree whose node set is $\mathscr{V}$ and whose edge set is a subset of $\mathscr{E}$. In this paper, we make the following assumption and use the properties of Laplacian matrices.

Assumption 1. The digraph $\mathscr{G}$ contains a spanning tree.

Lemma 2.1 (*Huang & Manton, 2010*). *Suppose that Assumption 1 holds. Then* (i) *$L_{\mathscr{G}}$ has a unique zero eigenvalue and all other $N-1$ eigenvalues have positive real parts.* (ii) *There exists a unique probability measure π^T such that $\pi^T L_{\mathscr{G}} = \mathbf{0}_{1\times N}$.* (iii) *There exists a nonsingular matrix $\Phi = (\mathbf{1}_{N\times 1}\ \phi_{N\times(N-1)})$ and $\Phi^{-1} = \begin{pmatrix} \pi^T \\ \psi_{(N-1)\times N} \end{pmatrix}$ such that $\Phi^{-1} L_{\mathscr{G}} \Phi = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{L}_{\mathscr{G}} \end{pmatrix}$. Here, all $N-1$ eigenvalues of $\tilde{L}_{\mathscr{G}}$, which are also nonzero eigenvalues of $L_{\mathscr{G}}$, have positive real parts.*

Hereinafter, we denote all distinct non-zero eigenvalues of $L_{\mathscr{G}}$ by $\lambda_1, \ldots, \lambda_l$, $\lambda_{\min} = \min\{Re(\lambda_m), 1 \le m \le l\}$, and $\lambda_{\max} = \max\{Re(\lambda_m), 1 \le m \le l\}$.

In this paper, we consider the consensus control for the N agents with the dynamics

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathscr{V}, \tag{1}$$

where $x_i(t) \in \mathscr{R}$ and $u_i(t) \in \mathscr{R}$ are the state and the control input of the ith agent. The measurement of the jth agent's state by the ith agent is modeled by

$$y_{ji}(t) = x_j(t) + \sigma_{ji}\eta_{ji}(t), \quad j \in \mathscr{N}_i, \tag{2}$$

where $\{\eta_{ji}(t), i \in \mathscr{V}, j \in \mathscr{N}_i\}$ are mutually independent standard white noises, and $\sigma_{ji} > 0$ is the noise intensity. Here, the noises are independent of the initial states. Denote $X(t) = [x_1(t), \ldots, x_N(t)]^T$, and $(\mathscr{G}, X)$ is called a dynamic network (Olfati-Saber & Murray, 2004). A measurement-based distributed protocol is a group of control inputs $\mathscr{U} = \{u_i, i \in \mathscr{V} | u_i(t) \in \sigma(x_i(s), y_{ji}(s); j \in \mathscr{N}_i, 0 \le s \le t), \forall\ t \ge 0\}$ (Li & Zhang, 2009). The agents are said to reach almost sure (*mean square, resp.*) strong consensus if there exists a random variable x^* such that, $\lim_{t\to\infty} x_i(t) = x^*, i \in \mathscr{V}$ almost surely ($\lim_{t\to\infty} E|x_i(t) - x^*|^2 = 0$ *and* $E|x_i(t)|^2 < \infty$, *resp.*). The agents are said to reach almost sure (*mean square, resp.*) weak consensus if for all distinct $i, j \in \mathscr{V}$, $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ almost surely ($\lim_{t\to\infty} E|x_i(t) - x_j(t)|^2 = 0$, *and* $E|x_i(t)|^2 < \infty$ *for all* $i \in \mathscr{V}$, *resp.*).

3. Asymptotic properties for the stochastic approximation type protocol

To attenuate the communication noises, the stochastic approximation type protocol $u_a(t) = [u_{a1}(t), \ldots, u_{aN}(t)]^T$ with

$$u_{ai}(t) = a(t) \sum_{j\in\mathscr{N}_i} a_{ij}(y_{ji}(t) - x_i(t)), \quad t \ge 0,\ i \in \mathscr{V}, \tag{3}$$

is a kind of effective consensus control laws. Here, the consensus gain function $a(\cdot) : [0, \infty) \to (0, \infty)$ is a piecewise continuous function. Substitute the protocol (3) into the system (1). Denote $X_a(t) = [x_{a1}(t), \ldots, x_{aN}(t)]^T$ as the corresponding closed-loop states with respect to $u_a(t)$, then the closed-loop system in the form of Itô stochastic differential equation is given by

$$dX_a(t) = -a(t)L_{\mathscr{G}}X_a(t)\,dt + a(t)\Sigma\,dw(t). \tag{4}$$

Here, $\Sigma = \mathrm{diag}(\alpha_1^T\Sigma_1, \ldots, \alpha_N^T\Sigma_N)$, where α_i is the ith row of the weighted adjacency matrix $\mathscr{A}$ and $\Sigma_i = \mathrm{diag}(\sigma_{1i}, \ldots, \sigma_{Ni})$ with $\sigma_{ji} = 0$ for $j \notin \mathscr{N}_i$. And $w(t) = (w_{11}(t), \ldots, w_{N1}(t), \ldots, w_{NN}(t))^T$ is an N^2 dimensional standard Brownian motion. In Li and Zhang (2009) and Wang and Zhang (2009), it was proved that if $\int_0^\infty a(t)dt = \infty$, $\int_0^\infty a^2(t)dt < \infty$, and Assumption 1 holds, then the stochastic approximation type protocol (3) ensures both mean square and almost sure strong consensus. Hereinafter, we denote $J = \mathbf{1}_{N\times 1}\pi^T$, and the consensus error $\delta_a(t) = (I_N - J)X_a(t)$. In this section, we will show how fast $\delta_a(t)$ vanishes in both senses of mean square and probability one. The following assumptions will be used.

Assumption 2. The consensus gain function $a(t)$ satisfies that $\lim_{t\to\infty} t^\gamma a(t)$ exists and is positive, where $\gamma \in (0.5, 1)$.

Assumption 3. The consensus gain function $a(t)$ satisfies that $\lim_{t\to\infty} ta(t) > 1/(2\lambda_{\min})$.

Remark 1. If $a(t)$ satisfies either Assumption 2 or Assumption 3, clearly, we have $\int_0^\infty a(s)ds = \infty$ and $\int_0^\infty a^2(s)ds < \infty$, which are typical conditions on the consensus gain function (Li & Zhang, 2009).

Firstly, we analyze the convergence rate of $E(\|\delta_a(t)\|^2)$.

Theorem 3.1. *Suppose that Assumption 1 holds. Apply the protocol (3) to the systems (1) and (2). If Assumption 2 holds, then the closed-loop system satisfies $E(\|\delta_a(t)\|^2) = O(t^{-\gamma})$; and if Assumption 3 holds, then $E(\|\delta_a(t)\|^2) = O(t^{-1})$.*

Proof. Noticing that $(I_N - J)L_{\mathscr{G}} = L_{\mathscr{G}} = L_{\mathscr{G}}(I_N - J)$, by Eq. (4), we have

$$d\delta_a(t) = -a(t)L_{\mathscr{G}}\delta_a(t)\,dt + a(t)(I_N - J)\Sigma\,d\omega(t). \tag{5}$$

Define $\tilde{\delta}_a(t) = \Phi^{-1}\delta_a(t)$, with Φ given in Lemma 2.1. Then it is sufficient to analyze the convergence rate of $E(\|\tilde{\delta}_a(t)\|^2)$ since Φ is nonsingular. By Eq. (5), Lemma 2.1 and $\pi^T\mathbf{1}_{N\times 1} = 1$, we have

$$d\tilde{\delta}_a(t) = -a(t)\mathrm{diag}(0, \tilde{L}_{\mathscr{G}})\tilde{\delta}_a(t)\,dt + a(t)\begin{pmatrix} \mathbf{0_1 \times N^2} \\ \psi_{(N-1)\times N}(I_N - J)\Sigma \end{pmatrix} d\omega(t).$$

Denote $\eta_a(t) = (\tilde{\delta}_{a2}(t), \ldots, \tilde{\delta}_{aN}(t))^T$. Then we get

$$\tilde{\delta}_{a1}(t) = \tilde{\delta}_{a1}(0) = \pi^T\delta_a(0) = \pi^T(I_N - J)X(0) = 0.$$

By the variation of constants formula for SDEs, we get

$$\eta_a(t) = \exp\Big(-\tilde{L}_{\mathscr{G}}\int_0^t a(s)\,ds\Big)\eta_a(0) + \int_0^t a(s) \times \exp\Big(-\tilde{L}_{\mathscr{G}}\int_s^t a(r)\,dr\Big)\psi_{(N-1)\times N}(I_N - J)\Sigma\,d\omega(s).$$

Hence, we know that $E(\|\tilde{\delta}_a(t)\|^2) = E(\|\eta_a(t)\|^2)$.

To obtain the desired upper bound, we begin by estimating $\exp\big(-\tilde{L}_{\mathscr{G}}\int_0^t a(s)\,ds\big)$. By Jordan matrix decomposition, there exists an invertible matrix P such that $P^{-1}\tilde{L}_{\mathscr{G}}P = J_{\tilde{L}_{\mathscr{G}}}$. Here, $J_{\tilde{L}_{\mathscr{G}}}$ is the Jordan normal form of $\tilde{L}_{\mathscr{G}}$, i.e. $J_{\tilde{L}_{\mathscr{G}}} = \mathrm{diag}(J_{\lambda_1,n_1}, J_{\lambda_2,n_2}, \ldots, J_{\lambda_q,n_q})$, where q is the number of Jordan blocks, $\lambda_1, \ldots, \lambda_q$ are all non-zero eigenvalues of $\tilde{L}_{\mathscr{G}}$, which may not be distinct from each other, and J_{λ_m,n_m} is the corresponding Jordan block of size n_m with eigenvalue λ_m, $m = 1, \ldots, q$. It follows that

$$\exp\Big(-\tilde{L}_{\mathscr{G}}\int_0^t a(s)\,ds\Big) = P\exp\Big(-J_{\tilde{L}_{\mathscr{G}}}\int_0^t a(s)\,ds\Big)P^{-1}$$

$$= P\,\mathrm{diag}\Big(\exp\Big(-J_{\lambda_1,n_1}\int_0^t a(s)\,ds\Big),\ldots,$$
$$\exp\Big(-J_{\lambda_q,n_q}\int_0^t a(s)\,ds\Big)\Big)P^{-1}$$

and

$$\exp\Big(-J_{\lambda_m,n_m}\int_0^t a(s)\,ds\Big)$$
$$=\begin{pmatrix} e^{-\lambda_m\int_0^t a(s)\,ds} & \cdots & \frac{\left(-\int_0^t a(s)ds\right)^{n_m-1} e^{-\lambda_m\int_0^t a(s)\,ds}}{(n_m-1)!} \\ 0 & \cdots & \frac{\left(-\int_0^t a(s)\,ds\right)^{n_m-2} e^{-\lambda_m\int_0^t a(s)\,ds}}{(n_m-2)!} \\ \vdots & \vdots & \vdots \\ 0 & \cdots & -\int_0^t a(s)\,ds e^{-\lambda_m\int_0^t a(s)\,ds} \\ 0 & \cdots & e^{-\lambda_m\int_0^t a(s)\,ds} \end{pmatrix}.$$

Note that the complex eigenvalues of the Laplacian matrix $L_{\mathcal{G}}$ will occur in conjugate pairs since $L_{\mathcal{G}}$ is real, thus, each element of $\exp(-\tilde{L}_{\mathcal{G}}\int_0^t a(s)\,ds)$ is a finite linear combination of

$$\Big(\int_0^t a(s)\,ds\Big)^k \cos\Big(Im(\lambda_m)\int_0^t a(s)\,ds\Big)e^{-Re(\lambda_m)\int_0^t a(s)\,ds}$$

and

$$\Big(\int_0^t a(s)\,ds\Big)^k \sin\Big(Im(\lambda_m)\int_0^t a(s)\,ds\Big)e^{-Re(\lambda_m)\int_0^t a(s)\,ds},$$

$k=0,\ldots,n_m-1,\ m=1,\ldots,q.$

Similarly, noticing $\psi_{(N-1)\times N}(I_N-J)\Sigma$ is a constant matrix, and each element of

$$\int_0^t a(s)\exp\Big(-\tilde{L}_{\mathcal{G}}\int_s^t a(r)\,dr\Big)\psi_{(N-1)\times N}(I_N-J)\Sigma\,d\omega(s)$$

is a finite linear combination of

$$\int_0^t \Big(\Big(\int_s^t a(r)\,dr\Big)^k \cos\Big(Im(\lambda_m)\int_s^t a(r)\,dr\Big) \times e^{-Re(\lambda_m)\int_s^t a(r)dr}a(s)d\omega_{ji}(s)$$

and

$$\int_0^t \Big(\Big(\int_s^t a(r)\,dr\Big)^k \sin\Big(Im(\lambda_m)\int_s^t a(r)\,dr\Big) \times e^{-Re(\lambda_m)\int_s^t a(r)dr}a(s)d\omega_{ji}(s),$$

$k=0,\ldots,n_m-1,\ m=1,\ldots,q,\ (j,i)\in\mathcal{E}$. It can be proved that for any fixed integer k, $Re(\lambda_m)>0$ and $s<t$, there exists $\varepsilon\in(0,Re(\lambda_m))$ and $C_1(k,\varepsilon)>0$, such that

$$\Big|\Big(\int_s^t a(r)\,dr\Big)^k \cos\Big(Im(\lambda_m)\int_s^t a(r)\,dr\Big)\Big| \le C_1 e^{\varepsilon\int_s^t a(r)\,dr}, \tag{6}$$

$$\Big|\Big(\int_s^t a(r)\,dr\Big)^k \sin\Big(Im(\lambda_m)\int_s^t a(r)\,dr\Big)\Big| \le C_1 e^{\varepsilon\int_s^t a(r)\,dr}. \tag{7}$$

For the case where Assumption 2 holds, by Lemmas A.1 and A.2, it follows that for any given $\lambda>0$, we have $e^{-2\lambda\int_0^t a(s)\,ds}=o(t^{-\gamma})$ and

$$E\Big(\int_0^t a(s)e^{-\lambda\int_s^t a(r)dr}\,d\omega_{ji}(s)\Big)^2=\Theta(t^{-\gamma}), \tag{8}$$

which together with (6) and (7) leads to $E(\|\eta_a(t)\|^2)=O(t^{-\gamma})$.

For the case where Assumption 3 holds, noticing that $\lim_{t\to\infty}(ta(t))>1/(2\lambda_{\min})$, we can always take ε satisfying $\lambda_{\min}-\varepsilon>1/(2\lim_{t\to\infty}(ta(t)))$. Then by inequalities (6) and (7) and Lemma A.3, we can conclude that $E(\|\eta_a(t)\|^2)=O(t^{-1})$. □

Theorem 3.1 gives upper bounds of convergence rate of the mean square consensus errors under general digraphs. In the following theorem, we get the exact mean square convergence rate of $\delta_a(t)$ if the network is balanced.

Assumption 4. The digraph $\mathcal{G}$ is balanced.

Theorem 3.2. *Suppose that Assumptions 1 and 4 hold. Apply the protocol (3) to the systems (1) and (2). If Assumption 2 holds, then the closed-loop system satisfies $E\|\delta_a(t)\|^2=\Theta(t^{-\gamma})$; and if Assumption 3 holds, then $E\|\delta_a(t)\|^2=\Theta(t^{-1})$.*

Proof. Under Assumptions 1 and 4, by (5) and the Itô formula, it is easy to check that

$$E(\|\delta_a(t)\|^2)\ge E(\|\delta_a(0)\|^2)e^{-2\lambda_N\int_0^t a(s)ds} + \mathrm{tr}((I-J)^2\Sigma\Sigma^T)\int_0^t a^2(s)e^{-2\lambda_N\int_s^t a(u)du}ds.$$

Here, λ_N is the largest eigenvalue of $\hat{L}_{\mathcal{G}}=(L_{\mathcal{G}}+L_{\mathcal{G}}^T)/2$, which is the Laplacian matrix of the symmetrized graph of $\mathcal{G}$ (Olfati-Saber & Murray, 2004). Noting the results obtained in Theorem 3.1, it will be sufficient to show $\int_0^t e^{-2\lambda_N\int_s^t a(u)du}a^2(s)ds=\Omega(t^{-\gamma})$ for $\gamma\in(0.5,1]$. And the result follows from Lemmas A.2 and A.3. □

If the network is undirected, we can give the upper bound of the convergence rate of $\delta_a(t)$ with probability one.

Assumption 5. The digraph $\mathcal{G}$ is an undirected graph.

Theorem 3.3. *Suppose that Assumptions 1 and 5 hold. Apply the protocol (3) to the systems (1) and (2). If Assumption 2 holds, then the closed-loop system satisfies $\|\delta_a(t)\|=O(t^{-\gamma/2}(\log\log b_{\lambda_{\max}}(t))^{1/2})$ a.s.; if Assumption 3 holds, then $\|\delta_a(t)\|=O(t^{-1/2}(\log\log b_{\lambda_{\max}}(t))^{1/2})$ a.s., where $b_{\lambda_{\max}}(t)=\int_0^t a^2(s)e^{2\lambda_{\max}\int_0^s a(r)\,dr}\,ds$.*

Proof. We first consider the case where Assumption 2 holds. Note that the adjacency matrix of an undirected graph is symmetric. Thus, the Laplacian matrix can be diagonalizable and all its eigenvalues are real. Under Assumptions 1 and 5, similar to the proof of Theorem 3.1, we can conclude that each element of $\delta_a(t)$ is a finite linear combination of $e^{-\lambda_m\int_0^t a(s)\,ds}$ and $\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)\,dr}\,d\omega_{ji}(s)$ for $m=1,\ldots,q$ and $(j,i)\in\mathcal{E}$. So it will be sufficient to analyze the convergence behaviors of these two kinds of items respectively. By Lemma A.1 and $\lim_{t\to\infty}b_{\lambda_{\max}}(t)=\infty$, we have

$$e^{-\lambda_m\int_0^t a(s)\,ds}=o\big(t^{-\gamma/2}(\log\log b_{\lambda_{\max}}(t))^{1/2}\big).$$

Hence, we only need to consider the convergence behaviors of $\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)\,dr}\,d\omega_{ji}(s)$. For any given $\lambda\in(0,\lambda_{\max})$, denote

$$b_\lambda(t)=\int_0^t a^2(s)e^{2\lambda\int_0^s a(r)\,dr}\,ds. \tag{9}$$

It is easy to verify that $\lim_{t\to\infty}b_\lambda(t)=\infty$. Thus, by the law of the iterated logarithm of stochastic integrals (Chen & Guo, 1991; Friedman, 1975), we have

$$\limsup_{t\to\infty}\frac{\left|\int_0^t a(s)e^{\lambda_m\int_0^s a(r)dr}\,d\omega_{ji}(s)\right|}{\left(2b_{\lambda_m}(t)\log\log b_{\lambda_m}(t)\right)^{1/2}}=1\quad \text{a.s.} \tag{10}$$

By (9), we know that

$$\left|t^{\gamma/2}(\log\log b_{\lambda_m}(t))^{-1/2}\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)\,dr}\,d\omega_{ji}(s)\right|$$
$$=\left(t^{\gamma}\int_0^t a^2(s)e^{-2\lambda_m\int_s^t a(r)dr}ds\right)^{1/2}$$
$$\times\frac{\left|\int_0^t a(s)e^{\lambda_m\int_0^s a(r)\,dr}\,d\omega_{ji}(s)\right|}{\left(b_{\lambda_m}(t)\log\log b_{\lambda_m}(t)\right)^{1/2}}.$$

Combining Lemma A.2 and (10), it is concluded that

$$\sup_{t\geq 0}\left|t^{\gamma/2}(\log\log b_{\lambda_m}(t))^{-1/2}\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)\,dr}\,d\omega_{ji}(s)\right|$$
$$<\infty\quad\text{a.s.}$$

Thus, we have

$$\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)\,dr}\,d\omega_{ji}(s)=O(t^{-\gamma/2}(\log\log b_{\lambda_m}(t))^{1/2})\quad\text{a.s.}$$

Noticing that $b_{\lambda_m}(t)\leq b_{\lambda_{\max}}(t)$ for $m=1,\ldots,q$, we get the conclusion of the theorem if Assumption 2 holds.

For the case where Assumption 3 holds, the result can be obtained similarly by using Lemma A.3. □

Remark 2. Noticing that $(\log\log 2b_{\lambda_{\max}}(t))^{1/2}=o(t^{\varepsilon})$, $\forall\,\varepsilon>0$, from Theorem 3.3, we know that $\|\delta_a(t)\|=O(t^{-\gamma/2+\varepsilon})$ a.s., which means that $\|\delta_a(t)\|$ is dominated by $O(t^{-\gamma/2+\varepsilon})$ asymptotically with probability one.

4. Kalman-Bucy filtering based consensus protocol

For the case with precise communication, i.e. $\sigma_{ji}=0$ for all $(j,i)\in\mathscr{E}$ (Olfati-Saber & Murray, 2004), the classical consensus protocol is $u_i(t)=\sum_{j\in\mathscr{N}_i}a_{ij}(x_j(t)-x_i(t))$, $t\geq 0$, $i\in\mathscr{V}$, i.e. each agent updates its state by the differences between its state and the information received from its neighbors. For the case with communication noises (i.e., $\sigma_{ji}>0$), since the ith agent cannot receive $x_j(t)$ from the jth agent accurately, it is natural to use an estimate instead of $x_j(t)$ itself. Here, based on the Kalman-Bucy filtering theory, we propose the following protocol

$$u_i(t)=\sum_{j\in\mathscr{N}_i}a_{ij}(\hat{x}_{j|i}(t)-x_i(t)),\quad t\geq 0,\ i\in\mathscr{V}. \tag{11}$$

Here, $\hat{x}_{j|i}(t)$ is the estimate of $x_j(t)$ by the ith agent based on its observations $\{y_{ji}(s), 0\leq s\leq t\}$, which satisfies

$$d\hat{x}_{j|i}(t)=u_j(t)\,dt+\frac{R_{ji}(t)}{\sigma_{ji}^2}\left(y_{ji}(t)-\hat{x}_{j|i}(t)\right)dt, \tag{12}$$

$$dR_{ji}(t)=-\frac{R_{ji}^2(t)}{\sigma_{ji}^2}\,dt, \tag{13}$$

where the initial values $\hat{x}_{j|i}(0)$ and $R_{ji}(0)>0$ are chosen arbitrarily.

Remark 3. For each channel $(j,i)\in\mathscr{E}$, by introducing $z_{ji}(t)=\int_0^t y_{ji}(s)ds$, from systems (1)–(2), we can extract a pair of state and measurement equations given by

$$dx_j(t)=u_j(t)dt, \tag{14}$$
$$dz_{ji}(t)=x_j(t)dt+\sigma_{ji}\,dw_{ji}(t),\quad z_{ji}(0)=0. \tag{15}$$

Noting that the state Eq. (14) has no process noise, we apply the Kalman-Bucy filter result (Kallianpur, 1980; Øksendal, 2010) to systems (14)–(15) and then Eqs. (12)–(13) are obtained. If $\hat{x}_{j|i}(0)=E(x_j(0))$, $R_{ji}(0)=var(x_j(0))$ and $x_j(0)$ is normally distributed and independent of $w_{ji}(t)$, then (12)–(13) constitute the classical Kalman-Bucy filter, i.e. $\hat{x}_{j|i}(t)$ is the conditional expectation minimizing the mean square error $E(\tilde{x}_{j|i}(t))^2$. However, since the statistical information about $x_j(0)$ is unknown, the initial values $\hat{x}_{j|i}(0)$ and $R_{ji}(0)>0$ will be chosen randomly in this paper.

Substituting the protocols (11)–(13) into the system (1), the closed-loop system is given by

$$dX(t)=(-L_{\mathscr{G}}X(t)+M\Upsilon(t))\,dt. \tag{16}$$

Here, $M=\text{diag}(\alpha_1,\ldots,\alpha_N)$ is an $N\times N^2$ dimensional block diagonal matrix, where α_i is the ith row of the adjacent matrix $\mathscr{A}$, $\Upsilon(t)=(\Upsilon_1^T(t),\ldots,\Upsilon_N^T(t))^T$ with $\Upsilon_i^T(t)=-(\tilde{x}_{1|i}(t),\ldots,\tilde{x}_{N|i}(t))^T$ being an $N^2\times 1$ vector, and $\tilde{x}_{j|i}(t)$ being the estimation error defined as

$$\tilde{x}_{j|i}(t)=\begin{cases}x_j(t)-\hat{x}_{j|i}(t), & \text{for } (j,i)\in\mathscr{E},\\ 0, & \text{otherwise}.\end{cases} \tag{17}$$

Similar to the case of stochastic approximation type protocols, we denote the consensus error $\delta(t)=(I_N-J)X(t)$. In this section, after giving the asymptotical properties of the estimate $\hat{x}_{j|i}(t)$ and the estimation error $\tilde{x}_{j|i}(t)$, we will show the convergence rate of $\delta(t)$ in both senses of mean square and probability one.

Lemma 4.1. ***Iterated logarithm law of estimation errors****: with randomly chosen initial values $\hat{x}_{j|i}(0)$ and $R_{ji}(0)>0$, the estimate $\hat{x}_{j|i}(t)$ satisfying Eqs. (12)–(13) is an asymptotically unbiased estimator of $x_j(t)$. Moreover, the estimation error $\tilde{x}_{j|i}(t)$ satisfies*

$$\limsup_{t\to\infty}\frac{|\tilde{x}_{j|i}(t)|}{\sqrt{2}\sigma_{ji}t^{-1/2}(\log\log t)^{1/2}}=1\quad\text{a.s.}$$

Proof. By Eq. (13), we get

$$R_{ji}(t)=\frac{R_{ji}(0)\sigma_{ji}^2}{\sigma_{ji}^2+R_{ji}(0)t}. \tag{18}$$

Combining the definition of the estimation error $\tilde{x}_{j|i}(t)$, Eqs. (1), (12), (15) and (18), it is easy to check that

$$d\tilde{x}_{j|i}(t)=-\frac{R_{ji}(0)}{\sigma_{ji}^2+R_{ji}(0)t}\tilde{x}_{j|i}(t)\,dt-\frac{R_{ji}(0)\sigma_{ji}}{\sigma_{ji}^2+R_{ji}(0)t}\,dw_{ji}(t),$$

which gives

$$\tilde{x}_{j|i}(t)=\frac{\sigma_{ji}^2}{\sigma_{ji}^2+R_{ji}(0)t}\tilde{x}_{j|i}(0)-\frac{R_{ji}(0)\sigma_{ji}}{\sigma_{ji}^2+R_{ji}(0)t}w_{ji}(t). \tag{19}$$

Noticing that $\tilde{x}_{j|i}(0)$ and $\omega_{ji}(t)$ are independent, we have

$$E(\tilde{x}_{j|i}(t))=\frac{\sigma_{ji}^2}{\sigma_{ji}^2+R_{ji}(0)t}E(\tilde{x}_{j|i}(0)),$$

and

$$var(\tilde{x}_{j|i}(t))=\frac{\sigma_{ji}^4 var(\tilde{x}_{j|i}(0))+R_{ji}^2(0)\sigma_{ji}^2 t}{(\sigma_{ji}^2+R_{ji}(0)t)^2}.$$

Although the initial values $\hat{x}_{j|i}(0)$ and $R_{ji}(0)>0$ are chosen randomly, we still have $E(\tilde{x}_{j|i}(t))=\Theta(t^{-1})$ and $var(\tilde{x}_{j|i}(t))=\Theta(t^{-1})$. Thus, $\hat{x}_{j|i}(t)$ is an asymptotically unbiased estimator of $x_j(t)$. Then by (19) and the law of iterated logarithm of Brownian motions, we have

$$\limsup_{t\to\infty} \frac{|\tilde{x}_{j|i}(t)|}{\sqrt{2}\sigma_{ji}t^{-1/2}(\log\log t)^{1/2}} = \limsup_{t\to\infty} \frac{R_{ji}(0)t}{\sigma_{ji}^2 + R_{ji}(0)t} \frac{|\omega_{ji}(t)|}{(2t\log\log t)^{1/2}} = 1 \quad \text{a.s.} \quad \square$$

Now, we will analyze the convergence rate of $E(\|\delta(t)\|^2)$ under Assumption 1.

Theorem 4.1. *Suppose that Assumption 1 holds. Apply the protocols (11)–(13) to the systems (1) and (2). Then the closed-loop system satisfies $E(\|\delta(t)\|^2) = O(t^{-1})$.*

Proof. Denote $\tilde{\delta}(t) = \Phi^{-1}\delta(t)$ with Φ given in Lemma 2.1. Then it is sufficient to show that $E(\|\tilde{\delta}(t)\|^2) = O(t^{-1})$ since Φ is nonsingular. By (16) and Lemma 2.1, we have

$$d\tilde{\delta}(t) = \left(-\text{diag}(0, \tilde{L}_{\mathcal{G}})\tilde{\delta}(t) + \begin{pmatrix} \mathbf{0}_{1\times N} \\ \psi_{(N-1)\times N}(I_N - J)M\Upsilon(t) \end{pmatrix}\right)dt.$$

Denote $\eta(t) = (\tilde{\delta}_2(t), \dots, \tilde{\delta}_N(t))^T$. Then we get

$$\tilde{\delta}_1(t) = \tilde{\delta}_1(0) = \pi^T\delta(0) = \pi^T(I_N - J)X(0) = 0,$$

$$\eta(t) = \exp\left(-\tilde{L}_{\mathcal{G}}t\right)\eta(0) + \int_0^t \exp\left(-\tilde{L}_{\mathcal{G}}(t-s)\right)\psi_{(N-1)\times N}(I_N - J)M\Upsilon(s)ds.$$

Subsequently, we have $E(\|\tilde{\delta}(t)\|^2) = E(\|\eta(t)\|^2)$. Following the computation of $\eta_a(t)$ in the proof of Theorem 3.1, we can conclude that each element of $\eta(t)$ is a finite linear combination of $t^k\cos(Im(\lambda_m)t)e^{-Re(\lambda_m)t}$, $t^k\sin(Im(\lambda_m)t)e^{-Re(\lambda_m)t}$, $\int_0^t (t-s)^k\cos(Im(\lambda_m)(t-s))\, e^{-Re(\lambda_m)(t-s)}\tilde{x}_{j|i}(s)\,ds$, and $\int_0^t (t-s)^k\sin(Im(\lambda_m)(t-s))\, e^{-Re(\lambda_m)(t-s)}\tilde{x}_{j|i}(s)\,ds$, $k = 0, \dots, n_m-1$, $m = 1, \dots, q$, $(i,j) \in \mathcal{E}$.

Notice that for any fixed integer k and $Re(\lambda_m) > 0$, there exists $\varepsilon \in (0, Re(\lambda_m))$ and $C_2(k, \varepsilon) > 0$, such that

$$\left|(t-s)^k\cos(Im(\lambda_m)(t-s))\right| \le C_2e^{\varepsilon(t-s)}, \tag{20}$$

$$\left|(t-s)^k\sin(Im(\lambda_m)(t-s))\right| \le C_2e^{\varepsilon(t-s)}. \tag{21}$$

By Lemmas B.1 and B.2, it follows that for any given $\lambda \in (0, \lambda_{\max})$ and $(j,i) \in \mathcal{E}$, we have

$$\int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s}ds = \Theta(t^{-1}),$$

and

$$E\left(\int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s}\omega_{ji}(s)\,ds\right)^2 = \Theta(t^{-1}).$$

This together with Eq. (19) leads to

$$E\left(\int_0^t e^{-\lambda(t-s)}\tilde{x}_{j|i}(s)\,ds\right)^2 = \Theta(t^{-1}), \quad \forall (j,i) \in \mathcal{E}. \tag{22}$$

Combining Eqs. (20), (21), (22) and $e^{-2\lambda t} = o(t^{-1})$, it is concluded that $E(\|\delta(t)\|^2) = O(t^{-1})$. $\square$

Theorem 4.1 gives the upper bound of the convergence rate of the mean square consensus error with general digraphs. If the network is undirected, we can get the exact mean square convergence rates of the consensus error.

Theorem 4.2. *Suppose that Assumptions 1 and 5 hold. Applying the protocols (11)–(13) to the systems (1) and (2), then the closed-loop system satisfies $E(\|\delta(t)\|^2) = \Theta(t^{-1})$.*

Proof. If Assumption 5 holds, then from the proof of Theorem 3.1, we can conclude that each element of $\delta(t)$ is a finite linear combination of $e^{-\lambda_m t}$ and $e^{-\lambda_m(t-s)}\tilde{x}_{j|i}(s)\,ds$, $m = 1, \dots, q$, $(i,j) \in \mathcal{E}$. Then the conclusion of the theorem follows from (22). $\square$

For the convergence rate of $\delta(t)$ with probability one, we have the following theorem for the case with general digraphs.

Theorem 4.3. *Suppose that Assumption 1 holds. Apply the protocols (11)–(13) to the systems (1) and (2). Then the closed-loop system satisfies $\|\delta(t)\| = O\left(t^{-1/2}(\log\log t)^{1/2}\right)$ a.s.*

Proof. Following the proof of Theorem 4.1, noting Eqs. (20), (21) and $e^{-\lambda t} = o\left(t^{-1/2}(\log\log t)^{1/2}\right)$ for any $\lambda > 0$, it will be sufficient to prove that for any given $\lambda \in (0, \lambda_{\max})$ and $(j,i) \in \mathcal{E}$, we have

$$\int_0^t e^{-\lambda(t-s)}\tilde{x}_{j|i}(s)\,ds = O\left(t^{-1/2}(\log\log t)^{1/2}\right) \quad \text{a.s.} \tag{23}$$

By Lemma 4.1, we know that for any given $\varepsilon > 0$, there exists $t_0 > 0$, such that for any given $s \ge t_0$, $|\tilde{x}_{j|i}(s)| \le (1+\varepsilon)\sqrt{2}\sigma_{ji}s^{-1/2}(\log\log s)^{1/2}$ a.s. Hence

$$\begin{aligned} &t^{1/2}(\log\log t)^{-1/2}\left|\int_0^t e^{-\lambda(t-s)}\tilde{x}_{j|i}(s)\,ds\right| \\ &\le t^{1/2}(\log\log t)^{-1/2}e^{-\lambda t}\left|\int_0^{t_0} e^{\lambda s}\tilde{x}_{j|i}(s)\,ds\right| \\ &\quad + (1+\varepsilon)\sqrt{2}\sigma_{ji}t^{1/2}(\log\log t)^{-1/2} \\ &\quad \times \int_{t_0}^t e^{-\lambda(t-s)}s^{-1/2}(\log\log s)^{1/2}\,ds \quad \text{a.s.} \end{aligned}$$

Notice that $\int_0^{t_0} e^{\lambda s}\tilde{x}_{j|i}(s)\,ds$ is bounded a.s. due to the continuous property of $\tilde{x}_{j|i}(s)$. Combining with

$$\lim_{t\to\infty} t^{1/2}(\log\log t)^{-1/2}e^{-\lambda t} = 0,$$

the first item tends to 0 as $t \to \infty$ a.s. For the second item, by L'Hôpital's rule, we have

$$\begin{aligned} &\lim_{t\to\infty} t^{1/2}(\log\log t)^{-1/2}\int_{t_0}^t e^{-\lambda(t-s)}s^{-1/2}(\log\log s)^{1/2}\,ds \\ &= \lim_{t\to\infty} \frac{d\left(\int_{t_0}^t e^{\lambda s}s^{-1/2}(\log\log s)^{1/2}\,ds\right)/dt}{d\left(e^{\lambda t}t^{-1/2}(\log\log t)^{1/2}\right)/dt} \\ &= 1/\lambda. \end{aligned}$$

Thus, it can be concluded that

$$\sup_{t\ge 0}\left(t^{1/2}(\log\log t)^{-1/2}\left|\int_0^t e^{-\lambda(t-s)}\tilde{x}_{j|i}(s)\,ds\right|\right) < \infty \quad \text{a.s.}$$

and (23) holds. $\square$

Now we apply the Kalman-Bucy filtering based protocols (11)–(13) to the leader-following scenario. Assume that agent 1 is the leader, and its state is ϑ, which is chosen randomly and remains a constant after. Since $\mathcal{N}_1 = \emptyset$, the first row of $\mathcal{A}$ is zero. The Laplacian matrix $L_{\mathcal{G}} = \begin{pmatrix} 0 & 0 \\ -B\mathbf{1}_{(\mathbf{n-1})\times\mathbf{1}} & L_{\bar{\mathcal{G}}} + B \end{pmatrix}$, where the digraph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}}\}$ denotes the subgraph formed by the $N-1$ followers, and $B = \text{diag}(a_{21}, \dots, a_{N1})$. Clearly, all eigenvalues of $L_{\bar{\mathcal{G}}} + B$, which are also nonzero eigenvalues of $L_{\mathcal{G}}$, have positive real parts. Denote $\bar{X}(t) = (x_2(t), \dots, x_N(t))$ and the tracking error $\bar{\delta}(t) = \bar{X}(t) - \vartheta\mathbf{1}_{(\mathbf{n-1})\times\mathbf{1}}$. In the following theorem, we will give the convergence rate analysis of the tracking error.

Theorem 4.4. *Suppose that Assumption 1 holds. Apply the protocols (11)–(13) to the systems (1) and (2). Then for this leader-following scenario, the closed-loop system satisfies* $E(\|\bar{\delta}(t)\|^2) = O(t^{-1})$ *and* $\|\bar{\delta}(t)\| = O\big(t^{-1/2}(\log\log t)^{1/2}\big)$ *a.s., which means that the protocols (11)–(13) ensure both mean square and almost sure strong consensus. Moreover, if the subgraph* $\bar{\mathscr{G}}$ *is undirected, then* $E(\|\bar{\delta}(t)\|^2) = \Theta(t^{-1})$.

Proof. By (16), for the leader-following case, we have

$$d\bar{X}(t) = (-(L_{\bar{\mathscr{G}}} + B)\bar{X}(t) + B\vartheta\mathbf{1}_{(\mathbf{n-1})\times\mathbf{1}} + \bar{M}\bar{\Upsilon}(t))dt.$$

Here, $\bar{M} = \mathrm{diag}(\alpha_2, \dots, \alpha_N)$, where α_i is the ith row of the adjacent matrix $\mathscr{A}$, and $\bar{\Upsilon}(t) = (\Upsilon_2^T(t), \dots, \Upsilon_N^T(t))^T$ with $\Upsilon_i^T(t) = -(\tilde{x}_{1|i}(t), \dots, \tilde{x}_{N|i}(t))^T$. Hence, the tracking error is given by $d\bar{\delta}(t) = (-(L_{\bar{\mathscr{G}}} + B)\bar{\delta}(t) + \bar{M}\bar{\Upsilon}(t))dt$, which gives

$$\bar{\delta}(t) = \exp\left(-(L_{\bar{\mathscr{G}}} + B)t\right)\delta(0) + \int_0^t \exp\left(-(L_{\bar{\mathscr{G}}} + B)(t-s)\right)\bar{M}\bar{\Upsilon}(s)ds.$$

By Theorems 4.1 and 4.3, we directly obtain that $E(\|\bar{\delta}(t)\|^2) = O(t^{-1})$ and $\|\bar{\delta}(t)\| = O\big(t^{-1/2}(\log\log t)^{1/2}\big)$ a.s. If the subgraph $\bar{\mathscr{G}}$ is undirected, then its Laplacian matrix $L_{\bar{\mathscr{G}}}$ is symmetric. And $L_{\bar{\mathscr{G}}} + B$ is also symmetric since B is a diagonal matrix. Therefore, $L_{\bar{\mathscr{G}}} + B$ can be diagonalizable and all its eigenvalues are real. Similar to the proof of Theorem 4.2, we have $E(\|\bar{\delta}(t)\|^2) = \Theta(t^{-1})$. □

Remark 4. Here, for the Kalman-Bucy filtering based protocol, we assume that the control input information of each agent was sent to its neighbors accurately. If the transmission of control inputs were corrupted by noises, then from the proofs of Lemma 4.1 and Theorem 4.1, there would be non-zero steady-state error for the mean square consensus error $E(\|\delta(t)\|^2)$, and the amplitude of $E(\|\delta(t)\|^2)$ would be proportional to the noise intensity. An interesting problem is whether the closed-loop consensus error is still vanishing if the control inputs of neighbors are neglected in (12). This might be true, however, without the control inputs of neighbors, the classical structure of the Kalman-Bucy filter is corrupted and the closed-loop analysis becomes more difficult.

5. Comparison of stochastic approximation type and Kalman-Bucy filtering based protocols

From Theorems 3.2 and 4.2, we can conclude that if Assumption 3 holds, then the closed-loop systems satisfy $E(\|\delta(t)\|^2) = \Theta\left(E(\|\delta_a(t)\|^2)\right)$. However, for other cases, the Kalman-Bucy filtering based protocols (11)–(13) may have a higher asymptotic convergence rate than the stochastic approximation type protocol (3). In this section, we will illustrate this property.

Theorem 5.1. *Suppose that Assumptions 1 and 5 hold. Apply the protocols (3) and (11)–(13) respectively to the systems (1) and (2). If* $\lim_{t\to\infty} ta(t)$ *exists and there exists a Laplacian eigenvalue* λ *satisfying that* $2\lambda\lim_{t\to\infty}(ta(t)) < 1$*, then the closed-loop systems satisfy* $E(\|\delta(t)\|^2) = o\left(E(\|\delta_a(t)\|^2)\right)$.

Proof. If Assumption 5 holds, from the proof of Theorem 3.1, we can conclude that each element of $\delta_a(t)$ is a finite linear combination of $e^{-\lambda_m\int_0^s a(r)\,dr}$ and $\int_0^t a(s)e^{-\lambda_m\int_s^t a(r)dr}\,d\omega_{ji}(s)$, $m = 1, \dots, q$. By the result of Theorem 4.2, it will be sufficient to show that $e^{-2\lambda\int_0^t a(s)ds} = \Omega(t^{-m})$ with $m < 1$. Note that $2\lambda\lim_{t\to\infty}(ta(t)) < 1$. Then we can take $\beta \in (\lim_{t\to\infty} ta(t), 1/2\lambda)$, and there exists $T_0 > 0$ such that $a(t) \le \beta t^{-1}$ for all $t \ge T_0$. Hence, we have

$$e^{-2\lambda\int_0^t a(s)ds} \ge e^{-2\lambda\int_0^{T_0} a(s)ds}T_0^{2\lambda\beta}t^{-2\lambda\beta}.$$

And the conclusion follows from $2\lambda\beta < 1$. □

Remark 5. From Theorem 4.2, we can see that the mean square convergence rate of the Kalman-Bucy filtering based protocol can always achieve $\Theta(t^{-1})$ for the case with undirected graphs. However, for the stochastic approximation type protocol, by Theorem 5.1, it is not always true that the mean square consensus error $E(\|\delta_a(t)\|^2) = \Theta(t^{-1})$ even if $a(t) = \Theta(t^{-1})$. The convergence rate of the stochastic approximation type protocol depends on not only the convergence rate of $a(t)$, but also the Laplacian eigenvalues of the network topology graph.

If the consensus gain function $a(t)$ satisfies Assumption 2, one can directly conclude that the closed-loop system satisfies $E(\|\delta(t)\|^2) = o\left(E(\|\delta_a(t)\|^2)\right)$ from Theorems 3.2 and 4.2, that is, the Kalman-Bucy filtering based protocol leads to a higher convergence rate than the stochastic approximation type protocol. Actually, the convergence rate of the closed-loop system under the Kalman-Bucy filtering based protocol is not slower than that under the stochastic approximation type protocol for both cases of $a(t) = \Theta(t^{-1})$ and $a(t) = \Theta(t^{-\gamma})$, $\gamma \in (0.5, 1)$.

A numerical example for the leader-following case will be given below.

Example 1. Consider a leader-following dynamic network of 4 agents with $\mathscr{V} = \{1, 2, 3, 4\}$. The weighted adjacency matrix of the communication graph is given by $a_{21} = a_{23} = a_{32} = a_{42} = a_{43} = 1$ and all other entries are zero. It is easy to check that $J_{L_{\mathscr{G}}} = \mathrm{diag}\big(0, 2, (3+\sqrt{5})/2, (3-\sqrt{5})/2\big)$. We take the state of the leader $x_1(t) \equiv 4$ and $X(0) = [4, 2, 5, 1]^T$, the noise intensities $\sigma_{12} = \sigma_{23} = \sigma_{32} = \sigma_{34} = \sigma_{24} = 0.7$. The stochastic approximation type protocol (3) and the Kalman-Bucy filtering based protocols (11)–(13) are implemented respectively. For the stochastic approximation type consensus protocol, we take the consensus gain function $a(t) = (t+1)^{-0.55}$. Fig. 1 shows the curves of closed-loop states and $tE(\|\delta_a(t)\|^2)$ under the protocol (3). Fig. 2 shows the curves of closed-loop states and $tE(\|\delta(t)\|^2)$ under the protocols (11)–(13). As shown in Figs. 1 and 2, under protocol (3) and protocols (11)–(13), the states of followers both converge to the leader's state. But as t increases, $tE(\|\delta(t)\|^2)$ converges, while $tE(\|\delta_a(t)\|^2)$ diverges. Hence, we can infer that $E(\|\delta(t)\|^2)$ vanishes faster than $E(\|\delta_a(t)\|^2)$.

Remark 6. For the protocols (11)–(13), one may wonder whether there exists a realization issue, since to calculate $u_i(t), u_j(t), j \in \mathscr{N}_i$, are needed. Below we give the discretization of the Kalman-Bucy filtering based protocols for the continuous-time systems, from which we can see that to calculate $u_i[kT_s]$, only $u_j[(k-1)T_s], j \in \mathscr{N}_i$, are used. Here, $u(t) = u(kT_s), t \in [kT_s, (k+1)T_s)$, for $k = 0, 1, \dots$, with T_s as the time sampling interval. For ease of notation, $u(kT_s)$ and $x(kT_s)$ are abbreviated as $u[k]$ and $x[k]$, respectively.

- System model and measurement model:

$$x_i[k+1] = x_i[k] + u_i[k]T_s, \quad \text{for } i \in \mathscr{V},$$
$$y_{ji}[k] = x_j[k] + \sigma_{ji}\eta_{ji}[k], \quad \text{for } j \in \mathscr{N}_i.$$

- Assumptions: (i) the ith agent can receive its neighbor's control input accurately. (ii) the noise sequences $\{\eta_{ji}(k),\ i \in \mathscr{V}, j \in \mathscr{N}_i\}$ are mutually independent stand white noises which are uncorrelated with x_0.
- Initialization: for all $i \in \mathscr{V}$ and $j \in \mathscr{N}_i$, we set $\hat{x}_{j|i}[0] = 1$, $R_{ji}[0] = 1$.

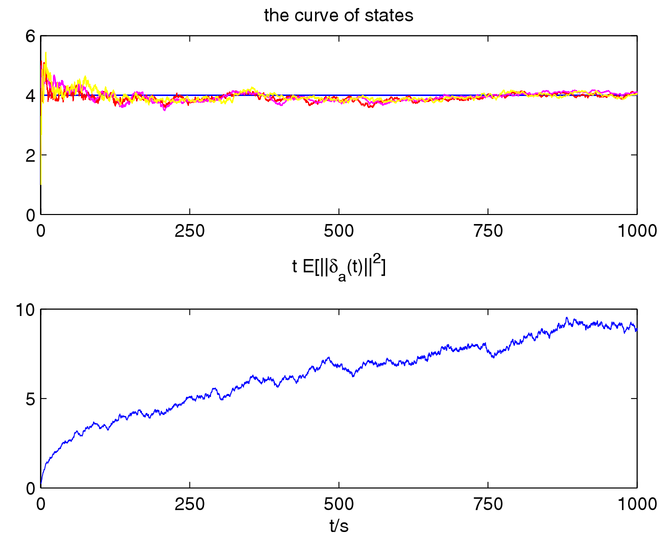

Fig. 1. Case with the stochastic approximation type protocol (3): curves of closed-loop states and $tE(\|\delta_a(t)\|^2)$.

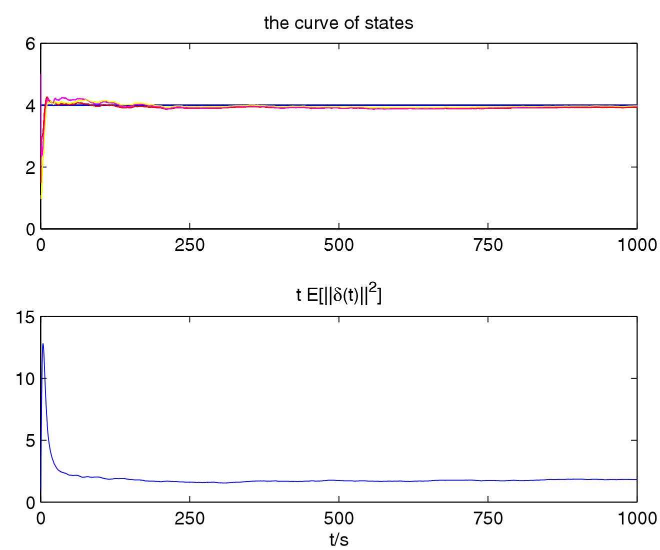

Fig. 2. Case with the Kalman-Bucy filtering based protocols (11)–(13): curves of closed-loop states and $tE(\|\delta(t)\|^2)$.

- For $k = 0, 1, 2, \ldots$, for all $i \in \mathscr{V}$, the ith agent takes the protocol
$$u_i[k] = \sum_{j \in \mathcal{N}_i} a_{ij}(\hat{x}_{j|i}[k] - x_i[k]),$$
and sends $u_i[k]$ and $x_i[k]$ to its neighbors.
- For all $i \in \mathscr{V}$, the ith agent updates its state as follows:
$$x_i[k+1] = x_i[k] + u_i[k]T_s.$$
- For all $i \in \mathscr{V}$, and $j \in \mathcal{N}_i$, we update the estimate of the neighbor's state and the Kalman gain for each communication channel given by
$$\hat{x}_{j|i}[k+1] = \hat{x}_{j|i}[k] + u_j[k]T_s + \frac{R_{ji}[k]}{\sigma_{ji}^2}\left(y_{ji}[k] - \hat{x}_{j|i}[k]\right)T_s,$$
$$R_{ji}[k+1] = R_{ji}[k] - \frac{R_{ji}^2[k]}{\sigma_{ji}^2}T_s.$$

Remark 7. Comparing with the stochastic approximation protocol, the Kalman-Bucy filtering based protocols (11)–(13) have superiority in the convergence rate. As a trade-off, more information are used, e.g., the statistic information of the noises, the accessibility of neighbors' inputs information. Besides, since the dynamics of the Kalman-Bucy filter is introduced in each agent, more computation is introduced. Here, as a preliminary research, we give a framework based on Kalman-Bucy filters and many interesting problems remain open for improving the performances with less information and resources.

6. Conclusion

In this paper, the continuous-time consensus problem of networked agents with noisy measurements and fixed directed topologies has been considered. By the tools of stochastic calculus, algebraic graph theory and limit analysis, we have analyzed the asymptotic properties of two kinds of consensus protocols: stochastic approximation type and Kalman-Bucy filtering based protocols.

For the stochastic approximation type protocol, we have quantified the convergence rate of the consensus error in both mean square and probability one. The relationship between the convergence rate and the consensus gain function has been revealed. It is found that if the consensus gain function $a(t)$ satisfies that $t^{\gamma}a(t)$ converges to a positive value as $t \to \infty$, $\gamma \in (0.5, 1]$, then the mean square of the consensus error is bounded above by $O(t^{-\gamma})$ asymptotically, and the consensus error is almost surely bounded above by $O(t^{-\gamma/2+\varepsilon})$, $\forall\, \varepsilon > 0$, asymptotically. For the Kalman-Bucy filtering based protocol, each agent uses Kalman-Bucy filters to get asymptotically unbiased estimates of neighbors' states and the control input is designed based on the protocol for precise communication and the certainty equivalence principle. The iterated logarithm law of estimation errors has been developed. It is shown that if the communication graph has a spanning tree, then the consensus error is bounded above by $O(t^{-1})$ in mean square and by $O(t^{-1/2}(\log\log t)^{1/2})$ almost surely. At last, we compare the stochastic approximation type protocol and the Kalman-Bucy filtering based protocol, and verify the superiority of the latter both theoretically and through simulation.

There are some restrictions in this paper and many interesting topics are still open.

(a) For the convergence rate analysis of the stochastic approximation type protocols, it is valuable to weaken Assumptions 2 and 3 to $\int_0^\infty a(s)ds = \infty$ and $\int_0^\infty a^2(s)ds < \infty$.
(b) It is of interest for models with dynamic topologies, and in particular, extending our results to networks with switching topologies.
(c) As stated in Remark 4, for the Kalman-Bucy filtering based protocol, we assume that the protocol input information of each agent is sent to its neighbors accurately. The problem whether the close-loop consensus error is still vanishing if neglecting neighbors' control inputs remains open.
(d) As stated in Remark 6, the Kalman-Bucy filtering based protocol has a discrete-time version. Extending the results in this paper to the discrete-time case is not difficult. An interesting topic is the sample-data based analysis for the overall hybrid system.
(e) As stated in Remark 7, more information and computation resources are required for implementing the Kalman-Bucy filtering based protocol. It would require more substantial investigation to improve this protocol and reduce the computational burden in future.

Appendix A. Auxiliary lemmas for Theorem 3.1

Lemma A.1. *Suppose that Assumption 2 holds. Then for any given $\lambda > 0$, we have $e^{-2\lambda \int_0^t a(s)\,ds} = o(t^{-\gamma})$.*

Proof. By Assumption 2, there exist $T_0 > 0$ and $\alpha > 0$ such that $a(t) \geq \alpha t^{-\gamma}$ for all $t \geq T_0$. Hence, it follows that

$$e^{-2\lambda \int_0^t a(s)\,ds} \leq e^{-2\lambda \int_0^{T_0} a(s)\,ds} e^{\frac{2\lambda\alpha}{1-\gamma} T_0^{1-\gamma}} \cdot e^{-\frac{2\lambda\alpha}{1-\gamma} t^{1-\gamma}}.$$

Take $u = t^{1-\gamma}$, then $u \to \infty$ as $t \to \infty$ by $\gamma \in (0.5, 1)$. By $\gamma/(1-\gamma) > 0$ and $-2\lambda\alpha/(1-\gamma) < 0$, we see that

$$\lim_{t\to\infty} t^\gamma e^{-\frac{2\lambda\alpha}{1-\gamma} t^{1-\gamma}} = \lim_{u\to\infty} u^{\frac{\gamma}{1-\gamma}} e^{-\frac{2\lambda\alpha}{1-\gamma} u} = 0.$$

Hence $\lim_{t\to\infty} t^\gamma e^{-2\lambda \int_0^t a(s)\,ds} = 0$, i.e. $e^{-2\lambda \int_0^t a(s)\,ds} = o(t^{-\gamma})$. □

Lemma A.2. *Suppose that Assumption 2 holds. Then for any given $\lambda > 0$, we have*

$$\lim_{t\to\infty} t^\gamma \int_0^t a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds = \lim_{t\to\infty} (t^\gamma a(t))/(2\lambda).$$

Proof. By Assumption 2 and Lemma A.1, it is easy to check that

$$\lim_{t\to\infty} t a^2(t) e^{2\lambda \int_0^t a(s)\,ds} = \lim_{t\to\infty} \left(t^{1-\gamma} \cdot (t^\gamma a(t))^2 \cdot t^{-\gamma} e^{2\lambda \int_0^t a(s)\,ds} \right) = \infty.$$

Then by the Cauchy criteria of improper integral, Lemma A.1 and Assumption 2, using L'Hôpital's rule, it follows that

$$\begin{aligned} &\lim_{t\to\infty} t^\gamma \int_0^t a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds \\ &= \lim_{t\to\infty} \frac{d\left(\int_0^t a^2(s) e^{2\lambda \int_0^s a(r)\,dr}\,ds\right)/dt}{d\left(t^{-\gamma} e^{2\lambda \int_0^t a(r)\,dr}\right)/dt} \\ &= \lim_{t\to\infty} (t^\gamma a(t))/(2\lambda). \quad \square \end{aligned}$$

Lemma A.3. *Suppose that Assumption 3 holds. Then for any given $\lambda > 0$ satisfying $2\lambda \lim_{t\to\infty}(ta(t)) > 1$, we have*

$$e^{-2\lambda \int_0^t a(s)\,ds} = o(t^{-1}) \tag{A.1}$$

and

$$\int_0^t a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds = \Theta(t^{-1}). \tag{A.2}$$

Proof. For any $\alpha \in (1/(2\lambda), \lim_{t\to\infty}(ta(t)))$ and $\beta > \lim_{t\to\infty}(ta(t))$, by Assumption 3, there exists $T_0 > 0$ such that $\alpha t^{-1} \leq a(t) \leq \beta t^{-1}$ for all $t \leq T_0$. Then we have

$$e^{-2\lambda \int_0^t a(s)\,ds} \leq e^{-2\lambda \int_0^{T_0} a(s)\,ds} \cdot T_0^{2\lambda\alpha} \cdot t^{-2\lambda\alpha}.$$

Note that $2\lambda\alpha > 1$ and hence Eq. (A.1) follows. Similarly, we get

$$\begin{aligned} &\int_0^t a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds \\ &\leq \int_0^{T_0} a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds + \beta^2 \int_{T_0}^t s^{-2} e^{-2\lambda\alpha \int_s^t r^{-1}\,dr}\,ds \end{aligned}$$

and

$$\int_0^t a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds \geq \alpha^2 \int_{T_0}^t s^{-2} e^{-2\lambda\beta \int_s^t r^{-1}\,dr}\,ds.$$

The item $\int_0^{T_0} a^2(s) e^{-2\lambda \int_s^t a(r)\,dr}\,ds$ is dominated by $o(t^{-1})$ from Eq. (A.1). Note $2\lambda\beta > 2\lambda\alpha > 1$, and for $m > 1$ we have

$$\int_{T_0}^t s^{-2} e^{-m \int_s^t r^{-1}\,dr}\,ds = 1/(m-1)(t^{-1} - t^{-m} T_0^{m-1}) = \Theta(t^{-1}).$$

Then Eq. (A.2) can be obtained. □

Appendix B. Auxiliary lemmas for Theorem 4.1

Lemma B.1. *Suppose that Assumption 1 holds. Apply the protocols (11)–(13) to the systems (1) and (2). Then for any given $\lambda > 0$, we have*

$$\lim_{t\to\infty} t \int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s}\,ds = \frac{1}{\lambda R_{ji}(0)}. \tag{B.1}$$

This lemma can be obtained by using L'Hôpital's rule and the details are omitted here.

Lemma B.2. *Suppose that Assumption 1 holds. Apply the protocols (11)–(13) to the systems (1) and (2). Then for any $\lambda > 0$, we have*

$$\lim_{t\to\infty} tE\left(\int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s} \omega_{ji}(s)\,ds\right)^2 = \frac{1}{\lambda^2 R_{ji}^2(0)}. \tag{B.2}$$

Proof. It is easy to check that

$$\begin{aligned} &E\left(\int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s} \omega_{ji}(s)\,ds\right)^2 \\ &= e^{-2\lambda t} \int_0^t \int_0^t \frac{e^{\lambda(s+u)} E\big(\omega_{ji}(s)\omega_{ji}(u)\big)}{\big(\sigma_{ji}^2 + R_{ji}(0)s\big)\big(\sigma_{ji}^2 + R_{ji}(0)u\big)}\,du\,ds \\ &= 2e^{-2\lambda t} \int_0^t \frac{e^{\lambda s}}{\sigma_{ji}^2 + R_{ji}(0)s} \int_0^s \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du\,ds. \end{aligned} \tag{B.3}$$

By $\lim_{u\to\infty} u \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du = \infty$ and the Cauchy criterion of the improper integral, we have

$$\lim_{s\to\infty} \int_0^s \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du = \infty,$$

which leads to

$$\lim_{s\to\infty} s \frac{e^{\lambda s}}{\sigma_{ji}^2 + R_{ji}(0)s} \int_0^s \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du = \infty.$$

Hence, by Eq. (B.3) and the Cauchy criterion of the improper integral, using L'Hôpital's rule twice, we have

$$\begin{aligned} &\lim_{t\to\infty} tE\left(\int_0^t \frac{e^{-\lambda(t-s)}}{\sigma_{ji}^2 + R_{ji}(0)s} \omega_{ji}(s)\,ds\right)^2 \\ &= \lim_{t\to\infty} \frac{d\left(2\int_0^t \frac{e^{\lambda s}}{\sigma_{ji}^2 + R_{ji}(0)s} \int_0^s \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du\,ds\right)/dt}{d\left(t^{-1} e^{2\lambda t}\right)/dt} \\ &= \lim_{t\to\infty} \frac{d\left(2\int_0^t \frac{ue^{\lambda u}}{\sigma_{ji}^2 + R_{ji}(0)u}\,du\right)/dt}{d\left(e^{\lambda t}\left(2\lambda R_{ji}(0) + (2\lambda\sigma_{ji}^2 - R_{ji}(0))t^{-1} - \sigma_{ji}^2 t^{-2}\right)\right)/dt} \\ &= 1/\left(\lambda^2 R_{ji}^2(0)\right), \end{aligned}$$

which gives (B.2). □

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